STATISTICS CANADA
CENSUS DIVISION

ANALYTICAL AND TECHNICAL MEMORANDUM

No. 7

A GOMPERTZ FIT THAT FITS:
APPLICATIONS TO CANADIAN FERTILITY PATTERNS

by

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Ottawa, April 1972

(Reproduced from **Demography** 9(1), February 1972)

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Résumé

Dans cette étude on essaie de raffiner la méthode pour ajuster la fonction de Gompertz aux taux de fécondité cumulés en utilisant des techniques itératives. Cette méthode est expérimentée avec la série des données historiques de la population canadienne. On évalue l'implication démographique des paramètres de la fonction de Gompertz ajustée à la distribution de la fécondité et on étudie l'utilité de la méthode projettant les tendances futures de la fécondité. La fonction de Makeham est aussi ajustée à la distribution de fécondité par la même technique et l'éfficacité relative de cette fonction est comparée à celle de Gompertz.

Abstract

In this paper an attempt is made to refine the method of fitting the Gompertz function to the cumulative fertility rates by using iterative techniques. The method is tested with the historical data series for the Canadian population. The demographic implication of the parameters of the Gompertz function as fitted to the fertility distribution is examined, and the usefulness of the method in projecting future fertility trends is studied. The Makeham function is also fitted to the fertility distribution by the same iterative technique, and the relative efficiency of this function is compared with that of the Gompertz.

* Grateful acknowledgement is made to the editor of Demography for permission to reproduce this paper.

1. Mathematical Functions in Fertility Studies

The use of mathematical functions to describe patterns of fertility behavior is a common demographic practice. Many demographic practitioners have studied the joint effect of fertility and mortality as represented by the net maternity function, and several curves have been investigated as graduations of this function. Among the best known studies (Keyfitz, 1968, pp. 141-169) are those using the Gaussian or normal curve (A.J. Lotka), the incomplete gamma function (Wicksell), and exponential curves (Hadwiger).

Others have attempted to find a graduation for schedules of age-specific fertility rates. These include Brass (1960), who used polynomial fittings, and Avery (1970), who compared several distributions as graduations of schedules of fertility rates.

A further series of investigations has considered the Gompertz function as a graduation of fertility distributions. This function has the form

$$Y = KA^{B^t} \qquad \text{.................................(1)}$$

It was developed as a description of mortality curve (Miller, 1946, pp. 42-47), but work by Martin (1967) and Wunsch (1966) has demonstrated that this function might provide a useful description of fertility distributions.

Martin fits the function by the method of selected points and Wunsch used a variation of this, partial totals, which in effect uses the mean value over selected ranges of the function. Selected point methods are not completely satisfactory, as will be discussed below, but despite this, both Martin and Wunsch achieved good fits to their data.

Romaniuk (1969) later applied these techniques to cumulative fertility rates for selected cohorts of Canadian women. He also obtained some good fits to the data, but he was primarily concerned with extrapolating the rates of cohorts with partially completed fertility as part of a projection exercise. The Gompertz function is not well suited for this purpose (see Section 5 below), as Romaniuk recognized in adopting another method, i.e., the chain ratios, for his extrapolations.

2. Method of Fitting that Minimizes the Sum of Squared Error

Although the selected points method can provide good fit to the data, there always remains an element of uncertainty in such a procedure. In discussing the use of the Gompertz function and the related Makeham function in mortality analysis, Miller (1946) notes:

> "...it is not advisable to determine the numerical values of the constants by the use of isolated points for, on account of the irregularity of the observed series, different sets of points produce widely varying values. There is no way of choosing four points beforehand with any assurance of obtaining a suitable representation of the data."

Fitting to a cumulative curve, as did Martin, reduces the problems arising from selected point fittings but does not eliminate them. Our goal is a method that incorporates all of the information available, and which minimizes the standard error measure, i.e., the sum of squared differences. Such least squares fits are straightforward for functions in which the constants enter linearly, but in the present case we must make recourse to an iterative procedure.

Keyfits (1968, pp. 214-215) presents a discussion of the descent method, which has been adopted here; we shall briefly summarize. The function is

$$Y(t) = KA^{B^t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Suppose P_i is the observed cumulative fertility at age i, i going from m to n, where m and n are the initial and final ages for which the observed fertility is available.

Let k_o, a_o, and b_o be an initial set of constants, and let $f_i(0)$ be the value of fertility at age i, estimated by substituting the initial set of constants in the Gompertz function. The sum of squares of deviation for the 0th iteration is given by

$$S_o = \sum_{i=m}^{n} (f_i(o) - P_i)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Now, let $k = k_o + h_1$, $a = a_o + h_2$, $b = b_o + h_3$ be the next approximations of the constants; if S_h is the sum of squares of deviation with this improved set, then by expanding S_h in the neighborhood of the initial set of constants k_o, a_o, b_o, by Taylor's theorem, we get

$$S_h = S_o + h_1 \frac{\partial S}{\partial k_o} + h_2 \frac{\partial S}{\partial a_o} + h_3 \frac{\partial S}{\partial b_o}$$

$$+ \frac{h_1^2}{2!} \cdot \frac{\partial^2 S}{\partial a_o^2} + h_1 h_2 \frac{\partial^2 S}{\partial a_o \partial b_o} + \text{etc.} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

where $\partial S/\partial k_o$ represents the value of the partial derivative of S with respect to k, with the values of the constants being taken as k_o, a_o, and b_o; and similarly for the other partial derivatives.

We will seek to minimize S_h; this involves deriving the partial derivatives of S_h with respect to k, a, and b and equating them to zero, thereby producing the correction factors h_1, h_2, and h_3. Thus

$$\frac{\partial S}{\partial k_o} + h_1 \frac{\partial^2 S}{\partial k_o^2} + h_2 \frac{\partial^2 S}{\partial k_o \partial a_o} + h_3 \frac{\partial^2 S}{\partial k_o \partial b_o} = 0$$

$$\frac{\partial S}{\partial a_o} + h_1 \frac{\partial^2 S}{\partial k_o \partial a_o} + h_2 \frac{\partial^2 S}{\partial a_o^2} + h_3 \frac{\partial^2 S}{\partial a_o \partial b_o} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

$$\frac{\partial S}{\partial b_o} + h_1 \frac{\partial^2 S}{\partial k_o \partial b_o} + h_2 \frac{\partial^2 S}{\partial a_o \partial b_o} + h_3 \frac{\partial^2 S}{\partial b_o^2} = 0$$

Writing in matrix form:

$$\begin{vmatrix} \frac{\partial S}{\partial k_o} \\ \frac{\partial S}{\partial a_o} \\ \frac{\partial S}{\partial b_o} \end{vmatrix} + \begin{vmatrix} \frac{\partial^2 S}{\partial k_o^2} & \frac{\partial^2 S}{\partial k_o \partial a_o} & \frac{\partial^2 S}{\partial k_o \partial b_o} \\ \frac{\partial^2 S}{\partial k_o \partial a_o} & \frac{\partial^2 S}{\partial a_o^2} & \frac{\partial^2 S}{\partial a_o \partial b_o} \\ \frac{\partial^2 S}{\partial b_o \partial k_o} & \frac{\partial^2 S}{\partial b_o \partial a_o} & \frac{\partial^2 S}{\partial b_o^2} \end{vmatrix} \cdot \begin{vmatrix} h_1 \\ h_2 \\ h_3 \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

or

$$F + (S.H.) = 0$$

$$H = -S^{-1}.F.$$

Thus correction factors can be found by premultiplying the vector of first partial derivatives by the inverse of the matrix of second partial derivatives. Initial values for the constants may be obtained by using the method of selected points, as suggested by Martin (1967). Following him, we make a transformation of the origin to t_o, thus

$$Y(t) = KA^{B^{(t-t_o)}}$$

Martin shows that

$$B = \left(\frac{\ell n\ Y(2) - \ell n\ Y(1)}{\ell n\ Y(1) - \ell n\ Y(0)}\right)^{\frac{1}{r}}$$

$$A = \mathrm{Exp}\left(\frac{\ell n\ Y(1) - \ell n\ Y(0)}{B^r - 1}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

$$K = \mathrm{Exp}\ (\ \ell n\ Y(0) - \ell n\ A)$$

where Y(2), Y(1), and Y(0) are the observed values at the selected age-points t(2), t(1), and t(0) respectively, and r is the age-distance between two consecutive selected points.

2.1 Data Fitted and the Computer Routine

All the arithmetical operations in the programmed routine were performed in double precision, and because of the computational convenience, the fertility rate per woman was considered instead of the rate per thousand women; the input data for the program were (a) the cumulative fertility rates; (b) the beginning and the ending ages of observed fertility; and (c) the selected age-points to start the process of iteration. Provision was also made in the program to extrapolate the fitted values up to age 49 if the observed rates were not available up to that point.

Data tested (Romaniuk, unpublished) were for the cohort rates for Canada from 1911 to 1945 and the period rates from 1926 to 1969. Almost invariably the convergence was reached in three to four iterations; the reduction in the sum of squares of deviation was impressive, resulting in the decrease in some cases of as much as 90 percent. As a rule, ten iterations were performed in each case. Table 1 gives some idea as to the efficiency of the method.

2.2 How Good is a Good Fit?

This decent method, then, provides a much improved fit in most cases; however, one must still face the question of how good, in demographic terms, is the fit. For example, the Gompertz curve fitted to the cumulative fertility rates of the 1911 cohort of Canadian women gives a sum of squared error of 0.04183; this represents an improvement of 93 percent over the curve fitted by selected points, but does not completely answer the question. To obtain an estimate of goodness of fit in demographic terms, we apply a demographic technique, standardization.

Aside from its possibilities of providing a succinct group of fertility parameters, a major use of the Gompertz function in fertility study might be in estimating the total number of children born to a cohort of women or during a period of time. This suggests that we measure the error that would be introduced into the estimate of total births by using the fitted instead of the actual rates.

As in all standardization exercises, the results permit two interpretations. One is that we are comparing the expected number of total births to a cohort, or during a period, with the births that would be produced if a population combined the given birth rates with the age composition of the standard population. The other interpretation is that we have chosen to weight the error in the fitted rates at each age by the "importance" of that age group, with importance being measured by the relative numbers of women in each age group in the standard population.

Along with the usual interpretations of standardization we face the usual problem, that of choosing a standard population. We reject the notion of using the actual population for each cohort or period, as such shifting standards would make comparisons unclear. What is desired are standard populations, one for the cohorts and one for the periods (which will be considered later) that yield an overall indication of the importance of each age group over the range of data in question.

For cohorts the choice ideally would be the stationary population from an appropriate cohort life table. The choice is not critical here, as mortality patterns in the ages 14 to 49 have not changed drastically over the period of the data. Under conditions of stable mortality, period and cohort life tables coincide, so a period life table is also appropriate as a standard. We have chosen as standard for the cohort comparisons the latest available official life tables for Canadian women, i.e., those for the period 1965-67 (Canada. Dominion Bureau of Statistics 1971a).

Neither is the choice critical for periods, as the Canadian age structure has not changed drastically during the periods covered by the data. Because it seemed desirable to choose a pattern representing the most recent age patterns, the standard population for periods was taken as the age distribution of Canadian women in the 1966 Census as graduated from five-year age groups using Sprague's multipliers (Canada. Dominion Bureau of Statistics, 1971b).

Thus armed with standardization and standard populations, we can attempt to answer the general question of how good, in demographic terms, are our fitted curves, and in particular, how good is the fit to the 1911 cohort.

We find that using the fitted, as opposed to the actual, rates would have produced an error of 2.80 percent in the age-standardized total births for the 1911 cohort. A three percent error is not overwhelming, and as we shall see most of the standardized errors are much lover. To give some terms of reference we could consider the simple approximation suggested by Keyfitz (1968, p. 350) to the sampling error of a standardized rate. Applying this technique to our data yields variances that are of the same order of magnitude as the net errors, such as 2.47 percent of the expected value for the 1911 cohort.

From another perspective, we can say that given only the three parameters of the Gompertz function, we can estimate the age-specific fertility rates of the 1911 cohort accurately enough so that the estimate of the standardized total births would be in error by only 2.80 percent. Not only is this error low; one can see from Table 2 that it is one of the poorer fits over the range of available Canadian data.

The usual problem with cohort data arises here in that the data for many of the cohorts are severely truncated. Data are complete up to age 45 for the cohorts of 1920 and earlier. Confining our attention to these cohorts, we see that

the net error ranges from 1.62 to 2.87 percent. This small error seems an acceptable cost to pay for reducing the information contained in 36 numbers, the rates by single years of age, to three parameters.

The present method yields fits that are best in the middle of the distribution; the fit in the tails is less satisfactory. Thus, the error of the fitted distribution is reduced as the cohort data are truncated. If we broaden the range of permissible data to include all those cohorts for which data through age 30 are available (in this case the cohorts of 1935 and earlier), we are still in the range that comprises 60 percent or more of the total fertility of the cohort. (This figure is based on the cohorts with completed fertility.) For these cohorts the fit is much better, ranging from 1.36 percent for the 1921 cohort to 0.00 percent (actually 0.0001 percent) for the 1931 cohort. Table 3 gives the complete distribution for the 1920 cohort as an example.

The better fit of these later cohorts represents a confounding of two effects. The first is the truncation effect, eliminating the tail of the distribution, where the curve doesn't fit so well anyhow; the second is the fact that the fit is intrinsically better with younger cohorts. This will be discussed in some detail below, but in brief it stems from the fact that as one moves from earlier to more recent cohorts, the fertility patterns of Canadian women move into a range in which the Gompertz function fits more efficiently.

We have been discussing the net error of these Gompertz fits. Table 1 also presents the gross error. This is the sum of the absolute value of the differences at each age. The net and gross error are defined as

$$\text{Net error} = \frac{\Sigma(r_i - \hat{r}_i)\ K_i}{\Sigma r_i k_i} \cdot 100 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

$$\text{Gross error} = \frac{\Sigma\left|r_i - \hat{r}_i\right|\ K_i}{\Sigma r_i k_i} \cdot 100 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

where r_i and $\hat{r}_i$ are the actual and estimated rates by age, respectively, and k_i is the standard population by age. The gross errors range, for cohorts of 1935 and before, between 2.5 and 14 percent. Which measure of error to consider as appropriate depends upon the intended use of the fitted curve. For those studies which refer to the entire cohort, such as parameterizing the fertility

experience or graduating the data, the net error seems the appropriate measure. In exercises which require the use of parts of the distribution in conjunction with other cohorts, such as translating the data for a number of cohorts into period data, the gross error would be appropriate.

Given these criteria, the fitted curves would be very good for the first type of exercise, but poorer for the second. Thus the Gompertz curve would not be well suited to what has now become almost the standard fertility projection technique - making estimates for cohorts and translating the results into period rates.

It should also be noted that the gross error measure assumes complete accuracy in the rates by single years of age, which is a questionable assumption. It may be that the graduated rates themselves are a better description of the underlying fertility process.

2.3 Fitting to Period Data

Analytical functions such as the Gompertz reflect a series of assumptions about the orderly workings of an underlying process. Schedules of real cohort rates represent the passage of a group of women through the ages of fertility, and it seems likely that they would have followed the kind of process underlying the analytical distribution. Thus cohort rates have seemed likely candidates for yielding close fits; they have seemed to be better candidates than period rates, which represent the experience of different groups of women.

However, to the extent that a given cohort of women is subjected to period phenomena which influence fertility - wars, depressions, and such - they are less likely to meet the conditions necessary for a close fit to the function. Martin (1967) makes this point in discussing his choice of the 1897 cohort of U.S. women for testing his method.

Schedules of period fertility rates also contain a fundamental order that is imposed by the strong physiological and social forces which arise from aging. Mortality analysis is largely founded upon this type of orderliness, but it has often received little emphasis in fertility analysis. In addition, period fertility rates, by definition, hold constant those period phenomena that disturb the cohort rates. Thus one might expect that period rates would fit our analytical function sbout as well as do cohort rates. Period rates also are free of the truncation problems which arise with cohort rates.

Table 4 presents the results of fitting the Gompertz function to period data for Canadian women from 1926 to 1969. (As was discussed above, the graduated age distribution for Canadian women, 1966, was chosen as the standard population.)

As the table indicates, the fits are at least comparable to those obtained for the cohorts. The net error ranges from 0.72 to 1.87 percent, gross error from 7.00 to 12.02 percent, and the sum of squared error from 0.0053 to 0,0417.

As with the cohort data, the fit improves as we move from the earlier to the more recent data. For the periods of the 1960's, the net error in estimated total births for the period is, save one exception, less than one percent. Since there are no truncation problems with period data, this steady improvement of the fit as one moves from earlier to recent periods is due largely to the fact that the more recent fertility patterns lie more comfortably in the usable range of the Gompertz function, as will be discussed in the next section. It should be noted, however, that the data themselves are smoother for the later periods, probably reflecting improved reporting.

For period distributions it is probably the net error that is the best measure of the efficiency of the fit. The principal uses of the fitted curves would be in projection, graduation, and parameterization, areas in which the net error seems most appropriate.

3. Demographic Interpretation of the Parameters

The ability of the Gompertz function to reproduce accurately, from three parameters, the age-specific fertility rates for these Canadian data suggests that we have before us a useable means of parameterizing the fertility experience. The next step in this process is to provide a demographic interpretation of the three parameters of the Gompertz function. We shall see that one of the advantages of this function is that a demographic interpretation is relatively straightforward.

As mentioned above, it is desirable to make a shift of origin on the age axis; for the iteration process used here, we have taken 24 as the origin. Thus we have taken 24 as the origin. Thus we have for the Gompertz function

$$Y = KA^{B^{(t-24)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (10)$$

where Y is the cumulative fertility rate by age, and t is age; what, to the demographer, are K, A, and B?

In the range that interests us, B will be less than one; thus as t increases without limit, $B^{(t-24)}$ approaches zero and Y approaches K. Thus K is the assymptote, the upper limit of Y, and in general can be interpreted as the total fertility rate of the cohort or period.

At the origin, X = 24, we have

$$Y = KA^{B^{o}} = KA.$$

Thus A is the proportion of the total fertility rate that has been attained by age 24, a common demographic measure often used to indicate how much of child-bearing is concentrated in the early years.

The remaining parameter B is not completely straightforward, but as is apparent from the curves of Graph 1, B is related to the degree to which age-specific fertility is concentrated about the peak age of fertility, to the moments of the fertility distribution.

The curves plotted in Graph 1 are the first derivatives of the Gompertz function

$$Z(x) = dY/dX = KA^{B^{(X-X_o)}} \ell nA^{B^{(X-X_o)}} \ell nB \quad \ldots\ldots\ldots\ldots (11)$$

When the Gompertz curve is interpreted as the cumulative fertility rates, the derivative represents the age-specific rate at exact age X. As Graph 1 demonstrates, these curves flatten out as B is increased while holding A and K constants. As B increases, the rates in the middle of the distribution decrease, while those at the tails increase. Thus the curve is very peaked for B = 0.7, very flat for B = 0.9.

To give some idea of the ranges and magnitudes involved we use some of the properties of the Gompertz function discussed by A.L. Titus in Appendix A. He shows that the curve of age-specific rates implied by the Gompertz fit must have the shape shown in Graph 1, and that a unique maximum exists, say at age X_m.

For present purposes, we take the origin at X_m and investigate the partial derivative of (15) with respect to B (to simplify the notation we write $t = X - X_m$).

$$\partial Z/\partial B = K \ln A \left[A^{B^t} t B^{t-1} \ln B \, B^t \ln A + A^{B^t} (B^{t-1} + \ln B t B^{t-1}) \right]$$

Collecting terms and using $H \ln G = \ln G^H$ gives

$$\partial Z/\partial B = (KA^{B^t}/B) \ln A^{B^t} \left[1 + t \ln B (\ln A^{B^t} + 1) \right] \text{.........(12)}$$

Since the sign of $\partial Z/\partial B$ will depend only upon the term in brakets, we let

$$F(B,t) = 1 + t \ln B (\ln A^{B^t} + 1)$$

We make use of another of the results shown by Titus (see Appendix A); he shows that within the range of constants appropriate here, the constant A must equal e^{-1} whenever the origin is fixed at the age of maximum fertility. Since we have chosen the origin here as at this maximum age, we have $A = e^{-1}$ and

$$F(B,t) = 1 + t \ln B (-B^t + 1) \text{ .... (13)}$$

We see at once that at the origin, the age of maximum fertility, $F(B,t) - 1$, and

$$\partial Z/\partial B = (KA^{B^t}/B) \ln A^{B^t} < 0$$

since $\ln A^{B^t}$ is always negative and the other terms positive. Thus at the origin, when B increases Z (t) decreases; in fact this will be the case whenever F(B,t) is positive.

Recalling that $0 < B < 1$, we see that

$$F(B,t) \leq 1 \text{ for all } t,$$

and furthermore that as

$$t \to \infty,\ F(B,t) \to -\infty$$
$$t \to -\infty,\ F(B,t) \to -\infty$$

and that $F(B,t)$ is monotonically decreasing with t to the right of the origin, monotonically decreasing to the left. Thus, $F(B,t) = 0$ and $\partial Z/\partial B = 0$ at two and only two values of t, once for t positive, once for t negative. These two roots, say t_1 and t_2 define a band of the $Z(t)$ curve. This band is roughly centered about the age of maximum fertility as the origin. Within this central band of t values, $Z(t)$ decreases as B increases, i.e., $\partial Z/\partial B$ is negative; for t to the left and right of this band, $Z(t)$ increases as B increases. Thus as B increases, the curve is flattened out and the variance of the curve is increased.

The three parameters of the Gompertz function, then, have clear demographic interpretations:

K: the total fertility rate

A: the proportion of total fertility completed at the origin, here at $t_o = 24$

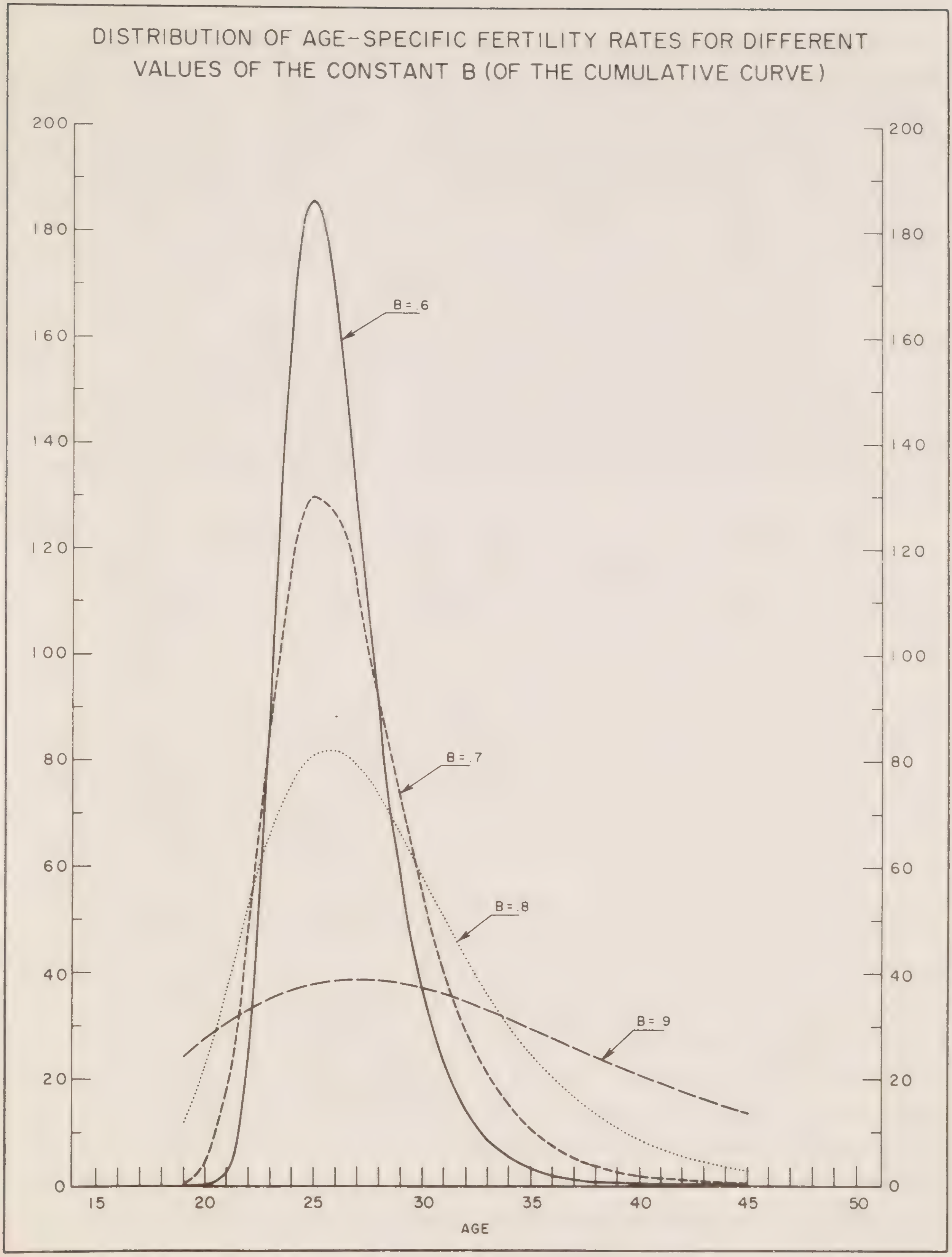
B: an indication of the variance of the distribution, of the spread of fertility over the age span.

We are now in a position to undertake for purposes of illustration a brief discussion of the changes in Canadian fertility patterns over the range of our data. But first we must discuss two properties of the Gompertz function which affect its use in fertility analysis.

The first of these is discussed by A.L. Titus in Appendix A. He discusses the first derivative of the function, which can be interpreted as the age-specific fertility rate at any instant. It shows that these rates increase monotonically to a maximum, and decrease monotonically after that. This seems reasonable, more so than his other point that at the age of maximum fertility, exactly e^{-1} or about 37 percent of the total fertility is completed for all sets of parameters within our range of interest. This is a curious property, but even more curious is the fact that, judging by the good fit of this function to the data, it is also a property of Canadian fertility patterns.

A second important property is apparent in Graph 1. The curves in this graph have been normalized to an upper assymptote of zero. We see that for relatively high values of B, the parameter controlling the spread of the distribution, the function does not approach the assymptote within the age range of human fertility. Table 5 presents some numerical values. A glance ahead to Tables 6 and 7 will demonstrate the importance of these figures. For most of the data presented, B falls within the range from 0.80 to 0.86.

GRAPH 1


DISTRIBUTION OF AGE-SPECIFIC FERTILITY RATES FOR DIFFERENT VALUES OF THE CONSTANT B (OF THE CUMULATIVE CURVE)
B = .6
B = .7
B = .8
B = 9
200
180
160
140
120
100
80
60
40
20
0
15
20
25
30
35
40
45
50
AGE

The fact that B decreases as one moves from earlier to more recent groups of women helps to explain why the fit of the curve is better for these more recent groups: the fit is better because the function is better behaved in this range. The behavior of the function for this range of B also helps to explain why attempts to achieve a better fit to the upper end of the distribution by fixing K at the total fertility rate and iterating only on the other two constants have not been successful.

4. The Changing Pattern of Canadian Fertility

To illustrate the use of this form of parameterization, we assume that all of the fits are equally tolerable and observe the trends in the constants. Table 6 presents the constants for the series of cohorts, Table 7 for the series of periods.

The most striking aspect of these series is that they are very orderly. For the cohorts, K, the equivalent of the total fertility rate, increases steadily from 2.89 for the 1911 cohort, peaks at 3.64 for the 1929 cohort, and declines steadily after that. The two parameters that summarize the timing of fertility A and B also show an orderly pattern of change. A, the proportion of fertility completed at age 24, remains roughly constant at about 24 percent from the 1911 to the 1915 cohort and increases steadily after that. B, the measure of the age spread of fertility, declines for the first five cohorts, reaches a plateau at about 0.847 through the 1919 cohort, and declines steadily after that.

Experiments indicate that truncated data sets tend to underestimate B and overestimate A, so one must exercise caution in interpreting the parameters for the more recent cohorts; however, one can state with some assurance that the pattern of cohort fertility has shown a trend toward births being concentrated in the younger ages along with a reduction in the variance of the birth rates by age, a squeezing together of the fertility distribution.

For the period data, there is no truncation problem, and one can use the entire series with some confidence. Here we see the same orderly movement of the parameters (Table 7). We find that K, the total fertility rate indicator, follows the well-known pattern: a decline from 1926 through the depression years of the 1930's, a moderate recovery during the war years, a strong recovery after 1945, peaking in the years from 1957 through 1959, and a very strong decline after that.

The two parameters of timing present a pattern of striking consistency. With one exception both series are nearly monotonic. The proportion of fertility completed by age 24, A, increases steadily from 23 percent in 1926 to 39 percent in 1969. The parameter measuring the spread of the fertility distribution, B, shows a steady decline, indicating a consistent tendency toward a squeezing together of the fertility rates by age. The exception to this steady change is a plateau in both series during the depression years of the 1930's; during this period A fluctuated around 23.5 percent and B around 0.861.

From this perspective, the much discussed boom-bust-war and boom cycle of the period between 1925 and 1960 seems to have had its principal disruptive effect on the total fertility rate of the periods involved. The timing parameters throughout the entire period were undergoing a very steady evolution toward younger and more concentrated fertility patterns. The depression halted this evolution of the timing patterns, but the process continued unabated through the war, the postwar baby boom, and the sharp decline of fertility in the 1960's.

The reader is referred to the error terms of Tables 3 and 4 for a reminder that these parameters do seem to provide a fairly accurate summary of the fertility experience.

5. Possible Use in Projections

A function that predicts total births accurately from a few parameters surely has implications for fertility projections. The fertility assumptions could be framed in terms of the parameters in the light of historical patterns, and the possibility is opened of linking these parameters to other data series such as economic projections.

The Gompertz function does appear promising for at least one strategy of fertility projection, which will be the subject of a later communication. However, it should be noted that the Gompertz function does not seem to appear well suited for use with the cohort-translation method of estimating fertility. This method makes its assumptions on fertility in terms of birth cohorts of women, and then translates schedules of cohort rates into period rates. It presents several difficulties (as do all methods), and the Gompertz function may not help in solving them.

The major difficulty with the cohort-translation method lies in translating the projected cohort rates into period rates. Usually the direct translation results in a series of period rates that may be unacceptable in the light of historical series. Ad hoc devices are usually employed to batter the period rates into shape, and the use of the Gompertz function may not offer immediate improvement

over these devices. Indeed, the high gross error of the estimates relative to the net error seems to indicate that caution should be exercised in the use of the Gompertz function for translation. However, further research may enable one to link explicitly the cohort parameters to those for the period, which will permit one to estimate the period parameters directly from those for cohorts.

Another possible use of a function such as the Gompertz might be to extrapolate the experience of cohorts of women with partially completed fertility. Again, this places us in an area where the function does not perform well. Experiments (to be reported in detail in the later communication on projection) indicate that truncated data sets do not yield accurate predictions of the final parameters; and even if they did, the function does not fit well at the tail end of the distribution. Table 8 indicates that even for those distributions which yield a very good fit to the data, the fit in the upper tail is relatively poor. In the next section, we discuss a related curve that does a better job in this respect at the cost of adding another parameter.

6. The Makeham Function

Although one hates to force another parameter into a process that already yields a good fit, completeness of discussion plus the relatively poor fit of the Gompertz function at the upper tail of the distribution suggests an investigation of a curve closely related to the Gompertz function, i.e., the Makeham curve.

The form of the Makeham curve is

$$Y = KS^{t} A^{B^{t}} \qquad \text{(14)}$$

This curve arises when one postulates a constant force of mortality (or fertility) operating along with that which varies with the size of the existing rate at a given age (Forsythe, 1924, pp. 67–68).

The procedure for fitting this curve by descent methods is essentially the same as for the Gompertz, except that there are now four parameters instead of three (see Section 2 above). Tables 9 and 10 summarize the results of fitting the Makeham function to selected Canadian data. The fit is indeed improved, as these tables indicate. Despite this improvement, we would not recommend the use of the Makeham over the Gompertz function, except perhaps for extrapolation of partially completed fertility.

One of our main goals is a succinct set of parameters, and moving from three to four is a real loss in compactness. Furthermore, the addition of the term S^X destroys the rather straightforward demographic interpretation of the parameters. Finally, except for the case of $S = 1$, when we are back to the Gompertz, the Makeham curve does not approach K symptotically, but increases without limit when S is greater than 1 or approaches zero when, as with the present data, S is less than one. For some of the early Canadian data the Makeham fit turned down before age 49, yielding cumulative rates that decline and implying negative age-specific rates.

7. Conclusion

As with most exploratory studies, this one has raised more questions than it has answered. We have shown that the Gompertz function, as fitted by the proposed method, yields good fits to a broad range of Canadian fertility patterns, and that these fits are close enough to suggest the three parameters of the Gompertz function, parameters which have clear demographic interpretations and are likely candidates for a parameterization of Canadian fertility patterns.

The next steps, currently underway, are a detailed investigation into the use of the proposed fitting method in Canadian projection work; the use of the function in the problem of translation of cohort into period fertility phenomena; a study of its range of application beyond Canadian data; and, if it does prove to have a fairly broad range of application, its use as a graduating function in analytical work.

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APPENDIX

Some Properties of the Gompertz Function

by

A.L. Titus

Some implications of fitting the Gompertz function to rates of cumulative fertility by age are revealed by examining the derivatives of the function. Given the function:

$$Y = KA^{B^{X-X_0}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

where K, A, B, and X_o are constants with $0 < A < 1$, $0 < B \quad 1$, and $X > 0$, then

$$dY/dX = K \ln A \ln B \, (A^{B^{X-X_o}} B^{X-X_o}) \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

and

$$d^2Y/dX^2 = (K \ln A \ln B)(A^{B^{X-X_o}} B^{X-X_o} \ln B)(B^{X-X_o} \ln A + 1) \ldots\ldots \quad (3)$$

If X is age, and Y is the cumulative fertility rate by age, then K is equivalent to the total fertility rate, and A is the fraction of K completed by age X_o; moreover, the first derivative of Y with respect to X, (equation 2), is the birth rate at exact age X, and the derivative of this function, i.e. the second derivative of the Gompertz function (equation 3), is the slope of the curve of instantaneous age-specific fertility rates.

By definition of the constants the product of the first two sets of terms in brackets in (3) is always negative. Thus the slope of (2) is determined by the sign of

$$B^{X-X_o} \ln A + 1) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

Since $0 < A < 1$ then $\ln A$ is negative. When in equation (4) the term $B^{X-X_o} \ln A$ is less than -1, the expression will be negative and the slope of equation (2) will be positive.

Similarly, when

$$B^{X-X_o} \ln A > -1, \text{ the slope of (2) will be negative}$$

and when

$$B^{X-X_o} \ln A = -1, \text{ (2) will be at a maximum.}$$

We note that at this maximum point, the age of the maximum age-specific fertility rate in the present interpretation

$$\ln A^{B^{X-X_o}} = -1$$

or

$$A^{B^{X-X_o}} = Y/K = e^{-1}$$

Thus fitting the Gompertz function to cumulative fertility rates implies that the curve of instantaneous age-specific rates will have the following properties:

(a) it will increase monotonically to a maximum and decrease monotonically after that,

(b) at the age of maximum fertility, exactly e^{-1}, about 37 per cent of the total fertility will have been completed, no matter what the values of K, A, B, and X_o.

The first property is as expected; the second is a surprising peculiarity.

TABLES 1 TO 10

TABLE 1. Distribution of Experimented Sets of Cohort and Period Fertility Rates by the Percent Reduction in the Sum of Squares of Deviation Due to Iteration

Percent Reduction in Sum of Square of Deviation	Cohorts *		Period *	
	Frequency	Per cent	Frequency	Per cent
90 +	7	18.9	-	-
75 to 90	8	21.6	12	26.7
50 to 75	15	40.6	18	40.0
40 to 50	7	18.9	8	17.8
Less than 40	-	-	7	15.5
Total	37	100.0	45	100.0

* Cohorts tested were for Canada for the years 1911-1946 and periods were for 1926 to 1969.

TABLE 2. The Final Sums of Squares, Net Error and Gross Error for Cohort Rates for Canada, 1911-1945

Year	Final SS	Net Error	Gross Error
1911	0.041826	2.80	13.76
1912	0.041708	2.79	13.25
1913	0.041133	2.87	12.07
1914	0.033225	2.49	11.09
1915	0.029024	2.38	10.48
1916	0.025934	2.28	9.97
1917	0.022754	2.21	9.50
1918	0.017063	2.04	9.40
1919	0.016748	1.89	9.05
1920	0.011885	1.66	8.33
1921	0.008389	1.36	7.49
1922	0.005827	1.02	6.87
1923	0.005965	0.76	6.16
1924	0.006116	0.54	5.81
1925	0.006140	0.33	5.83
1926	0.005045	0.17	5.19
1927	0.003760	0.14	4.40
1928	0.002468	0.16	4.07
1929	0.002237	0.11	3.61
1930	0.002323	0.09	3.73
1931	0.001686	0.00	3.10
1932	0.001209	0.02	2.74
1933	0.001160	0.18	3.61
1934	0.000730	0.21	2.88
1935	0.000403	0.15	2.42
1936	0.000270	0.14	2.06
1937	0.000135	0.08	1.52
1938	0.000126	0.20	1.82
1939	0.000096	0.23	1.78
1940	0.000034	0.11	1.20
1941	0.000029	0.11	1.33
1942	0.000012	0.07	0.99
1943	0.000010	0.09	1.08
1944	0.000004	0.01	1.10
1945	0.000001	0.05	0.96

TABLE 3. Actual and Fitted Values (Cumulative)
for the Canadian Cohort of 1920

Age	Actual	Estimated	Percent Diff.
14	0.0006	0.0027	- 377.5440
15	0.0031	0.0080	- 160.3258
16	0.0105	0.0203	- 92.7582
17	0.0307	0.0446	- 45.2292
18	0.0724	0.0869	- 20.1617
19	0.1455	0.1529	- 5.0923
20	0.2508	0.2466	1.6670
21	0.3879	0.3696	4.7096
22	0.5447	0.5207	4.4069
23	0.7090	0.6959	1.8437
24	0.8835	0.8898	- 0.7060
25	1.0889	1.0956	- 0.6082
26	1.3138	1.3066	0.5475
27	1.5162	1.5168	- 0.0368
28	1.7153	1.7210	- 0.3284
29	1.9002	1.9152	- 0.7919
30	2.0784	2.0968	- 0.8819
31	2.2401	2.2639	- 1.0603
32	2.3996	2.4157	- 0.6728
33	2.5453	2.5522	- 0.2740
34	2.6739	2.6739	0.0002
35	2.7892	2.7814	0.2798
36	2.8947	2.8758	0.6554
37	2.9837	2.9582	0.8539
38	3.0621	3.0298	1.0536
39	3.1245	3.0919	1.0457
40	3.1739	3.1454	0.9000
41	3.2084	3.1914	0.5312
42	3.2339	3.2309	0.0916
43	3.2505	3.2647	- 0.4377
44	3.2596	3.2937	- 1.0434
45	3.2641	3.3184	- 1.6627
46		3.3394	
47		3.3573	
48		3.3726	
49		3.3856	

The constants are A = 3.4581, B = 0.8467 and C = 0.2573

TABLE 4. The Final Sums of Squares, Net Error and Gross Error for Period Rates for Canada, 1926-1969

Year	Final SS	Net Error	Gross Error
1926	0.041696	1.86	12.02
1927	0.042620	1.87	12.73
1928	0.040895	1.85	13.09
1929	0.034042	1.73	11.80
1930	0.034385	1.69	11.88
1931	0.028241	1.65	10.18
1932	0.027092	1.70	11.16
1933	0.027170	1.80	11.52
1934	0.028290	1.88	11.71
1935	0.027231	1.86	11.48
1936	0.027788	1.89	11.82
1937	0.023837	1.80	11.22
1938	0.024061	1.71	11.94
1939	0.022343	1.67	11.67
1940	0.021919	1.54	11.40
1941	0.019407	1.40	10.41
1942	0.020463	1.42	10.72
1943	0.020810	1.44	10.40
1944	0.023691	1.55	10.57
1945	0.026097	1.57	10.81
1946	0.027418	1.35	10.27
1947	0.030079	1.23	10.44
1948	0.026075	1.23	10.10
1949	0.023668	1.24	9.46
1950	0.022809	1.26	9.28
1951	0.024752	1.25	9.76
1952	0.028168	1.23	10.20
1953	0.031159	1.21	10.68
1954	0.033256	1.15	10.86
1955	0.032883	1.11	10.51
1956	0.029107	1.04	9.57
1957	0.031558	1.01	9.72
1958	0.029678	1.00	9.84
1959	0.028585	0.95	9.62
1960	0.037584	1.08	13.20
1961	0.028168	0.89	9.64
1962	0.026963	0.88	9.80
1963	0.025980	0.89	9.95
1964	0.021279	0.89	9.23
1965	0.016754	0.91	9.24
1966	0.012176	0.83	8.95
1967	0.009615	0.77	8.55
1968	0.005409	0.72	7.00
1969	0.005259	0.73	7.00

TABLE 5. Percent of Total Fertility (K) Reached by Ages 45 and 49 for Different Values of B (A = 0.272)

B	Age 45	Age 49
.800	98.81	99.51
.825	97.73	98.94
.850	95.80	97.79
.875	92.42	95.48
.900	86.72	91.08

TABLE 6. The Parameters K, A, and B for Cohort Distributions for Canada, 1911-1945

Year	K	B	A
1911	2.8855	0.8593	0.2468
1912	2.9312	0.8569	0.2405
1913	3.0343	0.8541	0.2349
1914	3.0600	0.8501	0.2377
1915	3.0233	0.8485	0.2413
1916	3.0101	0.8471	0.2463
1917	3.0731	0.8476	0.2475
1918	3.0434	0.8473	0.2518
1919	3.4128	0.8478	0.2536
1920	3.4581	0.8467	0.2573
1921	3.4828	0.8456	0.2643
1922	3.4134	0.8438	0.2697
1923	3.4565	0.8423	0.2781
1924	3.4945	0.8411	0.2860
1925	3.4815	0.8400	0.2949
1926	3.4680	0.8390	0.3036
1927	3.5218	0.8367	0.3138
1928	3.4974	0.8333	0.3234
1929	3.6449	0.8311	0.3305
1930	3.6185	0.8268	0.3469
1931	3.5823	0.8212	0.3648
1932	3.4656	0.8171	0.3836
1933	3.3476	0.8135	0.3990
1934	3.2874	0.8073	0.4208
1935	3.2146	0.8018	0.4410
1936	3.0316	0.7943	0.4650
1937	2.9435	0.7870	0.4964
1938	2.8016	0.7797	0.5250
1939	2.6095	0.7726	0.5539
1940	2.4203	0.7645	0.5870
1941	2.1883	0.7590	0.6109
1942	1.9797	0.7523	0.6396
1943	1.6690	0.7442	0.6735
1944	1.4417	0.7436	0.6821
1945	1.3149	0.7397	0.6973

TABLE 7. The Parameters of K, A and B of Period Distributions for Canada, 1926-1969

Year	K	B	A
1926	3.5702	0.8641	0.2306
1927	3.5321	0.8646	0.2325
1928	3.5040	0.8642	0.2359
1929	3.4062	0.8623	0.2435
1930	3.4699	0.8614	0.2443
1931	3.3765	0.8597	0.2433
1932	3.2639	0.8610	0.2384
1933	3.0377	0.8617	0.2355
1934	2.9799	0.8622	0.2293
1935	2.9228	0.8617	0.2334
1936	2.8602	0.8614	0.2349
1937	2.7969	0.8601	0.2422
1938	2.8467	0.8592	0.2523
1939	2.7921	0.8584	0.2560
1940	2.8891	0.8557	0.2697
1941	2.9420	0.8527	0.2811
1942	3.0765	0.8520	0.2816
1943	3.1550	0.8505	0.2772
1944	3.1347	0.8528	0.2688
1945	3.1439	0.8541	0.2686
1946	3.4845	0.8493	0.2865
1947	3.6967	0.8466	0.3046
1948	3.5406	0.8474	0.3083
1949	3.5546	0.8465	0.3112
1950	3.5499	0.8461	0.3131
1951	3.5959	0.8460	0.3200
1952	3.7368	0.8450	0.3262
1953	3.8185	0.8446	0.3300
1954	3.9279	0.8446	0.3339
1955	3.9290	0.8437	0.3360
1956	3.9553	0.8420	0.3417
1957	4.0346	0.8420	0.3504
1958	3.9849	0.8407	0.3541
1959	4.0441	0.8396	0.3586
1960	4.0138	0.8393	0.3609
1961	3.9460	0.8379	0.3649
1962	3.8566	0.8363	0.3655
1963	3.7719	0.8359	0.3650
1964	3.6000	0.8361	0.3615
1965	3.2359	0.8372	0.3645
1966	2.8865	0.8362	0.3743
1967	2.6434	0.8333	0.3876
1968	2.4877	0.8294	0.3945
1969	2.4521	0.8294	0.3945

TABLE 8. The Percent Difference in the Upper Tail for Selected Cohort and Period Distributions

	Age	Actual	Estimated	Percent Diff
Cohort 1920	30	0.1783	0.1815	1.84
	40	0.0494	0.0535	8.31
	45	0.0044	0.00247	457.35
Period 1969	30	0.1089	0.1094	0.47
	40	0.0239	0.0223	6.49
	45	0.0031	0.0091	191.02

TABLE 9. Fitting of Makeham Function ($Y = AS^{x-24}C^{B^{x-24}}$) to Canadian (Cumulative) Cohort Fertility Rates, 1916-1939

Year	A	S	B	C	Error Measures: Sum of Squared Differences	Error Measures: Error in Estimated Births, Net	Error Measures: Error in Estimated Births, Gross
1916	3.8280	.99069	0.8701	0.1930	0.005055	0.05	6.16
1917	3.8554	.99101	0.8688	0.1964	0.006694	0.26	6.55
1918	3.7191	.99181	0.8661	0.2049	0.006374	0.40	6.99
1919	4.0864	.99245	0.8644	0.2106	0.007757	0.53	7.30
1920	3.9962	.99372	0.8602	0.2215	0.006647	0.62	7.16
1921	3.8502	.99547	0.8552	0.2380	0.006092	0.69	6.74
1922	3.5761	.99780	0.8484	0.2568	0.005392	0.72	6.50
1923	3.4391	1.0002	0.8418	0.2796	0.005960	0.79	6.18
1924	3.3277	1.0025	0.8360	0.3012	0.005713	0.83	5.83
1925	3.2426	1.0038	0.8326	0.3181	0.005420	0.73	5.87
1926	3.1793	1.0048	0.8299	0.3329	0.004156	0.62	5.25
1927	3.2310	1.0050	0.8278	0.3438	0.003013	0.55	4.50
1928	3.2405	1.0046	0.8255	0.3506	0.001996	0.50	3.96
1929	3.3195	1.0060	0.8215	0.3647	0.001625	0.49	3.32
1930	3.2365	1.0076	0.8148	0.3901	0.001568	0.51	3.60
1931	3.1856	1.0086	0.8083	0.4124	0.000968	0.43	2.96
1932	3.1048	1.0080	0.8059	0.4299	0.000800	0.36	2.86
1933	3.0548	1.0065	0.8050	0.4383	0.000992	0.42	3.44
1934	3.0157	1.0061	0.7998	0.4594	0.000631	0.40	2.69
1935	3.0056	1.0047	0.7964	0.4719	0.000367	0.27	2.40
1936	2.8152	1.0052	0.7887	0.5007	0.000245	0.25	2.14
1937	2.7991	1.0035	0.7835	0.5219	0.000128	0.14	1.56
1938	3.1289	.99242	0.7864	0.4705	0.000110	0.11	1.66
1939	3.3261	.98363	0.7855	0.4353	0.000062	0.09	1.51

TABLE 10. Fitting of Makeham Function ($Y = AS^{x-24}C^{B^{x-24}}$) to Canadian (Cumulative) Period Fertility Rates, 1931-1969

Year	A	S	B	C	Sum of Squared Differences	Error Measures: Error in Estimated Births, Net	Error Measures: Error in Estimated Births, Gross
1931	3.9987	.99364	0.8745	0.2046	0.018647	0.72	9.44
1932	3.9656	.99276	0.8773	0.1954	0.016291	0.67	10.08
1933	3.9006	.99082	0.8815	0.1826	0.013224	0.55	9.71
1934	3.9768	.98950	0.8841	0.1711	0.011965	0.49	9.64
1935	3.9057	.98942	0.8839	0.1738	0.010750	0.44	8.99
1936	3.8281	.98936	0.8838	0.1746	0.011660	0.45	9.36
1937	3.5519	.99114	0.8797	0.1896	0.011950	0.53	9.30
1938	3.4346	.99293	0.8756	0.2079	0.015339	0.64	10.13
1939	3.3083	.99357	0.8737	0.2148	0.015038	0.67	10.51
1940	3.2836	.99505	0.8682	0.2359	0.016466	0.72	10.67
1941	3.2323	.99628	0.8627	0.2546	0.015702	0.73	9.81
1942	3.4198	.99582	0.8631	0.2519	0.015277	0.66	9.76
1943	3.5522	.99531	0.8631	0.2448	0.013943	0.59	9.53
1944	3.6237	.99435	0.8673	0.2311	0.014922	0.58	9.64
1945	3.6272	.99445	0.8681	0.2314	0.017853	0.64	10.31
1946	3.7945	.99658	0.8588	0.2618	0.022572	0.71	10.01
1947	3.9127	.99768	0.8535	0.2866	0.027142	0.75	10.22
1948	3.7526	.99763	0.8545	0.2897	0.023264	0.74	9.90
1949	3.8074	.99719	0.8548	0.2890	0.019608	0.66	9.16
1950	3.8430	.99676	0.8557	0.2875	0.017376	0.59	8.64
1951	3.8722	.99696	0.8551	0.2954	0.019696	0.61	9.18
1952	3.9998	.99719	0.8535	0.3030	0.023266	0.62	9.75
1953	4.0665	.99739	0.8526	0.3081	0.026640	0.64	10.27
1954	4.1239	.99797	0.8509	0.3166	0.030290	0.70	10.63
1955	4.0882	.99834	0.8489	0.3217	0.030821	0.73	10.40
1956	4.0882	.99861	0.8465	0.3295	0.027545	0.71	9.47
1957	4.1419	.99889	0.8457	0.3404	0.030484	0.74	9.68
1958	4.0981	.99881	0.8446	0.3433	0.028428	0.72	9.72
1959	4.1341	.99906	0.8428	0.3499	0.027743	0.72	9.56

TABLE 10. Fitting of Makeham Function ($Y = AS^{x-24} C^{B^{x-24}}$) to Canadian (Cumulative) Period Fertility Rates, 1931-1969 - Concluded

Year	A	S	B	C	Sum of Squared Differences	Error Measures: Error in Estimated Births, Net	Error Measures: Error in Estimated Births, Gross
1960	4.0961	.99913	0.8422	0.3528	0.036867	0.86	13.20
1961	4.0013	.99940	0.8400	0.3593	0.027821	0.74	9.62
1962	3.9056	.99945	0.8382	0.3604	0.026681	0.74	9.84
1963	3.8339	.99929	0.8384	0.3584	0.025529	0.71	9.96
1964	3.6680	.99919	0.8390	0.3540	0.020750	0.68	9.24
1965	3.3082	.99905	0.8405	0.3556	0.016168	0.66	9.21
1966	2.9289	.99936	0.8384	0.3683	0.011956	0.67	9.01
1967	2.6719	.99952	0.8350	0.3829	0.009502	0.64	8.54
1968	2.5321	.99921	0.8324	0.3866	0.005106	0.50	6.82
1969	2.4959	.99921	0.8324	0.3866	0.004964	0.50	6.82